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Similarity solutions for fragmenting systems with continuous mass loss

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Abstract. A linear rate equation describing fragmentation processes, which includes both continuous and discrete loss of mass, is reduced to a partial differential equation. Different types of similarity solutions are obtained by applying Lie's similarity method.

1. Introduction

The fragmentation process is of considerable interest in several fields of physics. For an analytical treatment of fragmentation, continuous models are more appropriate. Fragmentation of continuous models has been discussed by McGrady and Ziff (1987), Cheng and Redner (1988) and Corngold and Williams (1989). Recently Edwards *et al* (1990) and Cai *et al* (1991) studied fragmentation processes which do not conserve solid mass, where the rate equation includes both continuous and discrete loss of solid mass.

Since continuous and discrete mass loss involve no collisions between particles and depend only on the interaction between each particle and its environment, assumed homogeneous, the rate equation for fragmentation with mass loss is linear. Huang *et al* (1991), used Laplace transformation, to give a series solution for fragmentation with continuous and discrete mass loss.

In work on similarity solutions Baumann *et al* (1991), for the first time in this field, discussed similarity solutions of the rate equation for discrete mass loss in the absence of continuous mass loss, where $\epsilon=0$ in (2). In this work the Lie similarity method is used to obtain solutions of the linear rate equation of Edwards *et al* (1990), which describe fragmentation with both continuous and discrete loss of mass.

2. Partial differential form of the rate equation

Following Huang *et al* (1992) and Edwards *et al* (1990), the linear rate equation which describes the evolution of the particle mass distribution $n(x, t)$ for a system of particles undergoing fragmentation with continuous mass loss is given by

$$\frac{\partial}{\partial t}n(x,t)=-a(x)n(x,t)+\int_x^\infty a(y)K(x,y)n(y,t)\,\mathrm{d}y+\frac{\partial}{\partial x}(c(x)n(x,t)) \tag{1}$$

where $a(x)$ is the fragmentation rate, $K(x, y)$ is the distribution of daughter particle mass x spawned by the fragmentation of a parent particle of mass y, and $c(x)$ is the continuous mass loss rate. The familiar rate equation, with discrete loss of mass, for polymer degradation considered by McGrady *et al* (1987) and Baumann *et al* (1991), follows from (1) by setting $c(x)=0$.

In the following we consider the power-law rates, which, as discussed by Edwards *et al* (1990) and Cai *et al* (1991), describe a wide spectrum of conditions for different values of power. Let $a(x)=x^{\alpha}$, $K(x,y)=g(y)x^{\nu}$ and $c(x)=\epsilon\, x^{\gamma}$, *with* $\epsilon\geqslant 0$. A normalization condition for $K(x,y)$

$$y-\int_0^y xK(x,y)\,\mathrm{d}x=\lambda y$$

allows for discrete mass loss during fragmentation events, with an ensemble averaged discrete loss fraction (Edwards *et al* 1990) satisfying $0\leqslant\lambda\leqslant 1$. Although λ may depend on y, the present work focuses on the implications of constant λ.

The normalization condition for $K(x,y)$, implies that

$$g(y)=2\varphi/y^{\nu+1}$$

with $\varphi=(1-\lambda)(\nu+2)/2$ and $\nu>-2$, so that (1) becomes

$$\frac{\partial}{\partial t}n(x,t)=-x^{\alpha}n(x,t)+2\varphi x^{\nu}\int_x^{\infty}y^{\alpha-\nu-1}n(y,t)\,\mathrm{d}y+\epsilon\frac{\partial}{\partial x}(x^{\gamma}n(x,t)). \qquad (2)$$

The physical conditions under which (2) is valid are also given in Huang *et al* (1991) for continuous mass loss ($\epsilon\neq 0$), and in Meesters and Ernst (1987), Cheng and Redner (1988) and Family *et al* (1986) for discrete mass loss ($\epsilon=0$).

Here we give a straightforward procedure to solve (2): if one multiplies (2) by a factor $x^{-\nu}$ and substitutes $w(u,t)=x^{-\nu}n(x,t)$ where $u=x^{\varphi}$, then (2) gives

$$\frac{\partial}{\partial t}w(u,t)=-u^{\beta}w(u,t)+(\delta+\mu)\eta u^{\delta-1}w(u,t)+\eta u^{\delta}\frac{\partial}{\partial u}w(u,t)$$
$$+2\int_u^{\infty}v^{\beta-1}w(v,t)\,\mathrm{d}v \qquad (3)$$

where

$$\beta=\alpha/\varphi \qquad \delta=1+(\gamma-1)/\varphi \qquad \eta=\epsilon\varphi \qquad \text{and} \qquad \mu=(\nu+\lambda(\nu+2))/2\varphi \qquad (4)$$

The corresponding partial differential equation to equation (3) is obtained by differentiating with respect to u

$$w_{tu}+Aw_{uu}+Bw_u+Cw=0 \qquad (5)$$

where

$$A=-\eta u^{\delta}$$
$$B=-\eta(2\delta+\mu)u^{\delta-1}+u^{\beta}$$
$$C=(\beta+2)u^{\beta-1}-\eta(\delta-1)(\delta+\mu)u^{\delta-2}. \qquad (6)$$

In the following we will give a complete solution to (5), in the case $\beta=\delta-1$, by using Lie's similarity method.

3. Similarity solution of the rate equation

To consider the symmetries of the linear partial differential equation (5), we have to examine a one-parameter (ε) group of infinitesimal transformations in u, t and w given by

$$\begin{aligned} \bar{u}&=u+U(u,t,w)+O(\varepsilon^2)\\ \bar{t}&=t+T(u,t,w)+O(\varepsilon^2)\\ \bar{w}&=w+W(u,t,w)+O(\varepsilon^2) \end{aligned} \tag{7}$$

and

$$\begin{aligned} \bar{w}_{\bar{u}}&=w_u+W^u+O(\varepsilon^2)\\ \bar{w}_{\bar{u}\bar{u}}&=w_{uu}+W^{uu}+O(\varepsilon^2)\\ \bar{w}_{\bar{t}\bar{u}}&=w_{tu}+W^{tu}+O(\varepsilon^2) \end{aligned} \tag{8}$$

where the functions W^u, W^{uu}, and W^{tu} in (8) are determined from (7) (Bluman and Kumei 1989). Invariance of (5) requires

$$\bar{w}_{\bar{t}\bar{u}}+A(\bar{u})\bar{w}_{\bar{u}\bar{u}}+B(\bar{u})\bar{w}_{\bar{u}}+C(\bar{u})\bar{w}=0. \tag{9}$$

By equations (7) and (8), to first order in ε, this becomes

$$W^{tu}+AW^{uu}+BW^u+CW+U\left(w_{uu}\frac{dA}{du}+w_u\frac{dB}{du}+w\frac{dC}{du}\right)=0 \tag{10}$$

which represents the determining equation for the group elements U, T, and W. Solving (10) for U, T, and W, one obtains:

$$\begin{aligned} T&=q_1+q_2t\\ U&=-q_2u/\beta\\ W&=q_3w. \end{aligned} \tag{11}$$

Thus we have obtained a one-parameter group of transformations depending on three arbitrary group constants q_1, q_2, and q_3.

The knowledge of the infinitesimal elements T, U, and W given in (11) enables us to construct three operators (for details see Bluman and Kumei 1989 and Olver 1986)

$$\begin{aligned} X_1&=\partial_t\\ X_2&=w\partial_w\\ X_3&=t\partial_t-\frac{u}{\beta}\partial_u \end{aligned} \tag{12}$$

where ∂_i is the partial differential operator.

Table 1. Lie algebra of the rate equation.

	X_1	X_2	X_3
X_1	0	0	X_1
X_2	0	0	0
X_3	$-X_1$	0	0

These three linear independent vector fields determine the symmetries under which (5) is invariant. Thus the corresponding Lie algebra of infinitesimal symmetries of (5) is spanned by the three vector fields X_1, X_2 and X_3. The commutation relations of these vector fields are given in table 1. Because a linear combination of the three vector fields determines the general symmetry of (5) we can use a combination of the vector fields to classify the types of solutions. Using the adjoint representation of the Lie algebra in table 2, we are able to distinguish four different types of solutions corresponding to the basic fields of an optimal system, given by X_1, X_3, X_1+X_2, and X_2+X_3.

In the following, we demonstrate that these combinations of symmetries produce essential types of solutions, by considering the similarity reductions obtained by solving the characteristic equations

$$\frac{\mathrm{d}t}{T}=\frac{\mathrm{d}u}{U}=\frac{\mathrm{d}w}{W}.$$

The general solution of these equations will involve two arbitrary constants, of which one constant takes the role of similarity variable, say s and the other constant, say $F(s)$, plays the role of similarity function. We mention that one obtains further solutions of (5) by applying finite group transformations to these solutions.

4. Group invariant solutions

In order to obtain the group invariant solutions, let us first consider the combination of $X_2+\tau X_3$, where τ is an arbitrary coefficient. The corresponding finite transformation reads

$$\bar{u}=u\,\mathrm{e}^{-\varepsilon\tau/\beta} \qquad \bar{t}=t\,\mathrm{e}^{\varepsilon\tau} \qquad \bar{w}=w\,\mathrm{e}^{\varepsilon} \tag{13}$$

where ε is the group parameter. The similarity variable s and the similarity solution $F(s)$ are

$$s=tu^{\beta} \qquad \text{and} \qquad w=t^{1/\tau}F(s). \tag{14}$$

Table 2. Adjoint representation of the Lie-algebra

	X_1	X_2	X_3
X_1	X_1	X_2	$X_3+\varepsilon\,\mathrm{e}^{-\varepsilon}X_1$
X_2	X_1	X_2	X_3
X_3	$\mathrm{e}^{\varepsilon/\beta}X_1$	X_2	X_3

Substitution of the similarity solution into (5) results in

$$\beta s(1-\eta\beta s)F_{ss}+[(1+1/\tau)\beta+\beta s(p-\eta(\beta-1))]F_s+qF=0 \tag{15}$$

where

$$\begin{aligned} p&=1-\eta(\mu+2\beta+2)\\ q&=(\beta+2)-\eta\beta(\beta+\mu+1). \end{aligned} \tag{16}$$

To reduce (15) to the standard form of hypergeometric equation we rescale s by $r=\eta\beta s$

$$r(l-r)F_{rr}+((1+1/\tau)+(p/\eta\beta+1/\beta-1)r)F_r+(q/\eta\beta^2)F=0. \tag{17}$$

Of course, there are only two linearly independent solutions so there are many relations between these special solutions (Murphy 1960). Here we give only the general solution

$$F(r)=C_1\,{}_2F_1(a,b,c,r)+C_2r^{1-c}{}_2F_1(1+a-c,1+b-c,2-c,r) \tag{18}$$

where C_1 and C_2 are arbitrary constants,

$$\begin{aligned} a&=-q/b\eta\beta^2\\ b&=[(\eta(1+\mu+2\beta)-1)\pm(1+\eta(6-2\mu)+\eta^2(\mu+1)^2)^{1/2}]/2\eta\beta\\ c&=1+1/\tau \end{aligned}$$

and $c\neq n$, n is an integer. The hypergeometric function is given by the series

$${}_2F_1(a,b,c,r)=1+(ab/c)r+(a(a+1)b(b+1))/(2!c(c+1))r^2+\dots. \tag{19}$$

If we invert all our previously used transformations applied to s and F as well, we obtain the complete solution to (5) for continuous loss of mass. We mention that $\epsilon=0$ will lead to the solution discussed by Baumann *et al* (1991) for discrete loss of mass, where (15) reduce to the confluent hypergeometric equation if we rescale $s=-z$

$$zF_{zz}+((1+1/\tau)-z)F_z-((\beta+2)/\beta)F=0. \tag{20}$$

The complete solution of (20) is given by

$$F(z)=C_1\,{}_1F_1(a,c;z)+C_2z^{1-c}{}_1F_1(a-c+1,2-c;z) \tag{21}$$

where C_1 and C_2 are arbitrary constants, $a=(\beta+2)/\beta$, $c=1+1/\tau$, and $c\neq n$, n is an integer. ${}_1F_1(a,c;z)$ is given by

$${}_1F_1(a,c;z)=1+(a/c)z+(a(a+1)/2!c(c+1))z^2+\dots. \tag{22}$$

Equation (20), with $\tau=1$, leads to the solutions discussed by McGrady and Ziff (1987) and Corngold and Williams (1989).

The hypergeometric type of solution of (5) is also obtained for the vector field X_3 including only scale invariance with respect to u and t. The corresponding similarity representation is given by

$$s=tu^{\beta}\qquad\text{and}\qquad w=F(s) \tag{23}$$

and the reduced equation (5) reads

$$\beta s(1-\beta\eta s)F_{ss}+(\beta-(\beta-1-p/\eta)\beta\eta s)F_s+qF=0. \tag{24}$$

A scaling of s, with $r=\beta\eta s$ gives the standard form of the hypergeometric equation:

$$r(1-r)F_{rr}+\left(1-\left(1-\frac{1}{\beta}-\frac{p}{\eta\beta}\right)r\right)F_r+\frac{q}{\eta\beta^2}F=0 \tag{25}$$

where p and q are as given in (16). This has a solution as in (18) with $c=1$ (for details see Murphy 1960).

For $\epsilon=0$, equation (24) reduce to the standard form of Kummer's equation by scaling s with $s=-z$

$$zF_{zz}+(1-z)F_z-(\beta+2)/\beta F=0. \tag{26}$$

Equation (26) is discussed by Baumann *et al* (1991) for discrete loss of mass.

Another type of similarity reduction can be obtained if we examine the linear combination $X_1+\tau X_2$. We introduce here the parameter τ to demonstrate that not only X_1+X_2 gives a similarity solution but also a linear combination with arbitrary coefficients.

The finite transformation for this combination can be written as

$$\bar{u}=u \qquad \bar{t}=t+\varepsilon\tau \qquad \bar{w}=e^{\epsilon}w.$$

The general reduction of this subgroup can be obtained by the similarity representation

$$s=u \qquad \text{and} \qquad w=e^{t/\tau}F(s)$$

The corresponding ordinary differential equation is

$$-\eta s^2F_{ss}+(p+1/\tau s^{-\beta})sF_s+qF=0. \tag{27}$$

A scaling of the similarity variable s by $s^{-\beta}=z$, transforms (27) to

$$z^2F_{zz}+(a_1+b_1z)zF_z+a_2F=0 \tag{28}$$

where

$$\begin{aligned} a_1 &= (\eta(\beta+1)+P)/\eta\beta \\ b_1 &= 1/\eta\beta\tau \\ a_2 &= -q/\eta\beta^2. \end{aligned} \tag{29}$$

Substitute $F=z^kU(z)$, where $k^2+(a_1-1)k+a_2=0$: equation (28) becomes

$$zU_{zz}+(A_1+b_1z)U_z+A_2U=0 \tag{30}$$

where

$$A_1=a_1+2k \qquad \text{and} \qquad A_2=b_1k.$$

To reduce (30) to the standard form of the confluent hypergeometric equation we rescale z by $z=-r/b_1$:

$$rU_{rr}+(A_1-r)U_r-(A_2/b_1)U=0. \tag{31}$$

The complete solution of (31) is given by (21), where $a=A_2/b_1$, and $c=A_1$.

For discrete loss of mass, equation (27) becomes

$$(1+s^{-\beta}/\tau)sF_s+(\beta+2)F=0. \tag{32}$$

Substitute $z=s^{-\beta}$, then (32) has the solution

$$F(z)=C_1(z/z+\tau)^{(\beta+2)/\beta} \tag{33}$$

with C_1 as a constant of integration.

Inverting all previously used transformations, we obtain

$$w(u,t)=e^{t/\tau}(1+\tau u^{\beta})^{-1-2/\beta} \tag{34}$$

as a solution of (5) for arbitrary parameter τ and $\epsilon=0$.

The last vector field of our optimal system which remains to be discussed is X_1. Since we have here a translational symmetry in time, it is easy to get the similarity representation by $s=u$ and $w=F(s)$. The reduced equation for this class follows from (5) to be

$$-\eta s^2 F_{ss}+pF_s+qF=0. \tag{35}$$

Equation (35) has the solution

$$F(s)=C_1 s^{y1}+C_2 s^{y2} \tag{36}$$

where C_1 and C_2 are constants of integration, and $y1$ and $y2$ are given by;

$$y1, y2=\tfrac{1}{2}(1+p/\eta)\pm[(1+p/\eta)^2+4q/\eta]^{1/2}. \tag{37}$$

We mention that, for the discrete loss of mass, equation (35) will lead to the solution discussed by Baumann *et al* (1991), where

$$F(s)=Cs^{-(\beta+2)} \tag{38}$$

with C as a constant of integration.

By applying similarity analysis we are able to classify three types of solutions for equation (5). Two subgroups reduce to the hypergeometric equation, one to the confluent hypergeometric equation, and the other to spatial scaling.

5. Conclusions

We have demonstrated that by performing Lie's similarity method to partial differential equation (5) resulting from integro-differential equation (1), a great variety of solutions are obtained by using group transformation. Some of these solutions are discussed for the initial value problem by Huang *et al* (1991) using Laplace transformation. Similarity methods deliver these special type of solution and several others for fragmentation processes with continuous mass loss of different types. Special cases of our results, for discrete loss of mass, are close to that discussed by Baumann *et al* (1991).

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